# NOTES

## *Flow through Regular Polygonal Channels*

### INTRODUCTION

In an earlier communication,[1] we presented a general treatment of flow in rectangular channels from a square to a wide slit. We now put forward a method for obtaining flow equations for regular polygonal channels (both parallel-sided and tapered) on the assumed approximation that the flow rate through a regular polygonal section equals that through a circle of equal cross-sectional area with radius $R$. In the process we are able to obtain an approximate Rabinowitsch correction which applies to regular polygonal channels. In deriving output equations for an equilateral triangular channel and for a square channel by using the natural geometry of the systems without recourse to the circular approximation, we intend to subject these alternatives to experimental scrutiny and will report on the results in due course.

### UNTAPERED POLYGONAL CHANNELS

Let $N$ be the number of sides of length $S$ of a regular polygon, $\alpha$ the half-angle subtended by $S$ with respect to any concentric circle, so that $\alpha = \pi/N$, and $R$ the radius of a circle the area of which is equal to the area of the polygon. From elementary geometry,

$$\text{area}_{\text{polygon}} = \frac{S^2}{4} N \cot \frac{\pi}{N} = \pi R^2 \tag{1}$$

and

$$R = \frac{S}{2}\sqrt{\frac{N}{\pi} \cot \frac{\pi}{N}} = \xi_N S \tag{2}$$

so that

$$\text{area}_{\text{polygon}} = \pi S^2 \xi_N^2$$

where

$$\xi_N^2 = \frac{N}{4\pi} \cot \frac{\pi}{N} \tag{3}$$

It is appreciated that the assumption of equal output for equal cross-sectional areas involving a regular polygon on the one hand and a circle on the other assumes that the isovels show approximately the same pattern. This is certainly true when $N$ becomes large enough for the angle $\alpha$ to approach zero. In the triangle, $\alpha$ is 60°; in the square, 45°; in the pentagon, 36°; and in the dodecagon, 15°. But since the isovels very closely resemble those of a circle once $N \geqslant 5$, it is felt that the approximation is justified, and we therefore proceed with our assumption.

The Rabinowitsch-corrected Hagen–Poiseuille equation for the cylindrical channel of cross-sectional area equal to that of the polygon is

$$Q = \frac{\pi n}{3n + 1} \left(\frac{\Delta P}{2\eta L}\right)^{1/n} R^{(3n+1)/n} \tag{4}$$

From the balance of forces in the polygonal (prismatic) channel

$$\tau \cdot N \cdot SL = \Delta P \pi \xi^2 S^2$$

or

$$\tau = \frac{\Delta P S}{4L} \cot \frac{\pi}{N} \tag{5}$$

But $\tau = \eta \dot{\gamma}^n$, and using eq. (5),


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$$\left(\frac{\Delta P}{2\eta L}\right)^{1/n} = \dot{\gamma}\left(\frac{2}{S\cot(\pi/N)}\right)^{1/n} \tag{6}$$

Substitution in (4) gives

$$Q = \frac{\pi n}{3n+1}\dot{\gamma}\left(\frac{2}{S\cot(\pi/N)}\right)^{1/n} R^{(3n+1)/n}$$

And since R is given by eq. (2),

$$Q = \frac{\pi n}{3n+1}\dot{\gamma}\left(\frac{2}{S\cot(\pi/N)}\right)^{1/n}\left(\frac{S}{2}\sqrt{\frac{N}{\pi}\cot\frac{\pi}{N}}\right)^{(3n+1)/n} \tag{7}$$

Therefore,

$$\dot{\gamma} = \left(Q\cdot\frac{3n+1}{\pi n}\right)\left(\frac{S\cot(\pi/N)}{2}\right)^{1/n}\left(\frac{S}{2}\sqrt{\frac{N}{\pi}\cot\frac{\pi}{N}}\right)^{-(3n+1)/n}$$

or

$$\dot{\gamma} = \frac{Q}{\pi}\left(\frac{3n+1}{n}\right)\left(\frac{S}{2}\right)^{-3}\left(\cot\frac{\pi}{N}\right)^{(1-3n)/2n}\left(\frac{N}{\pi}\right)^{-(3n+1)/2n} \tag{8}$$

Now, when $n = 1$ (the Newtonian case),

$$\dot{\gamma} = \dot{\gamma}_N = \frac{Q}{\pi}\cdot 4\cdot\frac{8}{S^3}\left(\cot\frac{\pi}{N}\right)^{-1}\left(\frac{N}{\pi}\right)^{-2}$$

or

$$\dot{\gamma}_N = \frac{32Q\pi\tan(\pi/N)}{S^3N^2} \tag{9}$$

and on dividing (8) by (9) and simplifying it is seen that

$$\frac{\dot{\gamma}}{\dot{\gamma}_N} = \frac{3n+1}{4n}\left(\sqrt{\frac{N}{\pi}\tan\frac{\pi}{N}}\right)^{(n-1)/n} \tag{10}$$

the approximated Rabinowitsch correction for regular polygons.

Checking eq. (10), it is seen that $\dot{\gamma}/\dot{\gamma}_N = 1$ when $n \to 1$, whilst, for $N \to \infty$, $\pi/N$ becomes a small angle, so that

$$\tan\frac{\pi}{N} \to \frac{\pi}{N}$$

and

$$\frac{\dot{\gamma}}{\dot{\gamma}_N} \to \frac{3n+1}{4n}\left(\frac{N}{\pi}\cdot\frac{\pi}{N}\right)^{(n-1)/2n} = \frac{3n+1}{4n}$$

Having thus obtained a working Rabinowitsch correction for regular polygons we proceed as usual:

Using (5) and the power law,

$$\frac{\Delta PS}{4L}\cot\frac{\pi}{N} = \eta\left[\frac{4Q}{\pi R^3}\cdot\frac{3n+1}{4n}\left(\sqrt{\frac{N}{\pi}\tan\frac{\pi}{N}}\right)^{(n-1)/n}\right]^n = \eta\left(\frac{Q}{\pi R^3}\cdot\frac{3n+1}{n}\right)^n\left(\sqrt{\frac{N}{\pi}\tan\frac{\pi}{N}}\right)^{n-1}$$

Substituting for $S$ on the left hand side as per (2), we then get

$$\frac{2\Delta PR\cot(\pi/N)}{4L\sqrt{(N/\pi)\cot(\pi/N)}} = \text{r.h.s. of above equation}$$

or

$$\frac{\Delta PR}{2L\sqrt{(N/\pi)\tan(\pi/N)}} = \text{r.h.s.}$$

and making $\Delta P$ the subject:

$$\Delta P = 2\eta L\left(Q\cdot\frac{3n+1}{\pi n}\cdot\sqrt{\frac{N}{\pi}\tan\frac{\pi}{N}}\right)^n\cdot R^{-(3n+1)} \tag{11}$$

TABLE I
$\lambda^n$ as a Function of $N$ and $n$

| $N$ | $\frac{N}{\pi}$ | $\frac{\pi}{N}$ | $\tan\frac{\pi}{N}$ | $\lambda^n$ $n = 1$ | $n = 0.75$ | $n = 0.55$ | $n = 0.40$ | $n = 0.33$ | $n = 0.25$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.955 | 60 | 1.732 | 1.286 | 1.208 | 1.109 | 1.042 | 1.014 | 1.003 |
| 4 | 1.273 | 45 | 1.000 | 1.128 | 1.095 | 1.051 | 1.020 | 1.007 | 1.002 |
| 5 | 1.591 | 36 | 0.727 | 1.076 | 1.056 | 1.031 | 1.012 | 1.004 | 1.001 |
| 6 | 1.910 | 30 | 0.577 | 1.050 | 1.037 | 1.020 | 1.008 | 1.003 | 1.001 |
| 7 | 2.228 | 25.71 | 0.482 | 1.036 | 1.027 | 1.015 | 1.006 | 1.002 | 1.000 |
| 8 | 2.546 | 22.5 | 0.414 | 1.027 | 1.020 | 1.011 | 1.004 | 1.001 | 1.000 |
| 9 | 2.864 | 20 | 0.364 | 1.021 | 1.016 | 1.009 | 1.003 | 1.001 | 1.000 |
| 10 | 3.183 | 18 | 0.325 | 0.017 | 1.013 | 1.007 | 1.003 | 1.001 | 1.000 |
| 11 | 3.501 | 16.36 | 0.294 | 1.014 | 1.010 | 1.006 | 1.002 | 1.001 | 1.000 |
| 12 | 3.820 | 15 | 0.268 | 1.012 | 1.009 | 1.005 | 1.002 | 1.001 | 1.000 |
| 13 | 4.137 | 13.85 | 0.246 | 1.009 | 1.007 | 1.004 | 1.001 | 1.000 | 1.000 |
| 14 | 4.456 | 12.86 | 0.228 | 1.008 | 1.006 | 1.005 | 1.001 | 1.000 | 1.000 |
| 15 | 4.774 | 12 | 0.213 | 1.008 | 1.006 | 1.003 | 1.001 | 1.000 | 1.000 |

## TAPERING CHANNEL WITH POLYGONAL CROSS SECTION

To adapt eq. (11) for a tapering cross section we follow the usual technique of considering the incremental pressure drop $dP$ over a corresponding length, the latter being expressed in terms of $dr$ and the taper angle $\theta$: $dl = -dr \cot\theta$. This is then followed by integration between the entrance and exit radius to recover the overall pressure drop $\Delta P$ and results in

$$\Delta P = \frac{2\eta \cot\theta}{3n}\left(Q \cdot \frac{3n+1}{\pi n} \cdot \sqrt{\frac{N}{\pi}\tan\frac{\pi}{N}}\right)^n (R_2^{-3n} - R_1^{-3n}) \tag{12}$$

where $R_1$ and $R_2$ can, of course, always be expressed in terms of $S_1$ and $S_2$. Now, it is already known that for a tapering *circular* section we have

$$\Delta P = \frac{2\eta \cot\theta}{3n}\left(Q \cdot \frac{3n+1}{\pi n}\right)^n (R_2^{-3n} - \mathrm{R}_1^{-3n}) \tag{13}$$

Inspecting eq. (12) and (13) it is seen that

$$\frac{\Delta P_{\text{polygon}}}{\Delta P_{\text{circle}}} = \left(\sqrt{\frac{N}{\pi}\tan\frac{\pi}{N}}\right)^n = \lambda^n \tag{14}$$

$\lambda^n$ represents the increase in pressure to maintain a given flow rate through a tapering die of polygonal cross section when compared with a circular cross section *of identical cross sectional area* and identical length. The values for $N/\pi$ (in radians), $\pi/N$ (in degrees), and $\tan(\pi/N)$ are given in Table I for polygons with $N$ ranging from 3 to 15. Table I also gives the appropriate values for $\lambda^n$ with $n$ ranging from 0.25 to unity.

It is seen that the "error" involved in the approximation as quantified by $\lambda^n$ *for Newtonians* ($n = 1$) is as much as 29% for the triangle, 13% for the square, and 8% for the pentagon, but thereupon falls rapidly as $N$ increases beyond 5.

For pseudoplastics, however, this error reduces substantially as $n$ reaches typical values for polymer melts and at $n = 0.33$, for example, the error even in the triangle is a mere 1.4%. This suggests that the approximation should be of considerable practical value.

In the next section we shall consider an alternative treatment of triangular channels which is not dependent upon the assumptions made earlier on, but which is based upon the specific geometry involved.

## THE PRESSURE DROP IN EQUILATERAL TRIANGULAR PARALLEL-SIDED AND TAPERING DIES

From the balance of forces in this geometry, $\tau \cdot 3SL = \Delta P \cdot (S^2/4)\sqrt{3}$, so that

$$\tau = \frac{\Delta P\, S\sqrt{3}}{12L} \tag{15}$$

According to F. N. Cogswell,[2] the "best guess" for the shear rate is

$$\dot{\gamma}_N = \frac{Q\sqrt{3}}{2S^3\sqrt{2}} \tag{16}$$

Cogswell also considers that the Rabinowitsch correction for this geometry is reasonably approximated by that which applied to circular ducts, namely:

$$\dot{\gamma} = \frac{3n+1}{4n}\dot{\gamma}_N = \frac{3n+1}{8n}\sqrt{\frac{3}{2}}\frac{Q}{S^3}$$

According to the Power law, therefore,

$$\eta = \frac{\tau}{\dot{\gamma}^n} = \frac{\Delta P\, S}{4\sqrt{3}L}\left(\frac{8n}{3n+1}\sqrt{\frac{2}{3}}\frac{S^3}{Q}\right)^n$$

Making $\Delta P$ the subject of the equation,

$$\Delta P = \frac{4\eta L\sqrt{3}}{S}\left(\frac{8n}{3n+1}\sqrt{\frac{2}{3}}\frac{S^3}{Q}\right)^{-n}$$

which simplifies to

$$\Delta P = 4\eta L\left(Q\cdot\frac{3n+1}{8n\sqrt{2}}\right)^n\left(\sqrt{3}\right)^{1-n} S^{-(3n+1)} \tag{17}$$

In the Newtonian case ($n = 1$),

$$\Delta P = (\Delta P)_N = LQ\eta\sqrt{2}S^{-4} \tag{18}$$

while for $n = 1/3$ (a common value for many polymer melts),

$$\Delta P = (\Delta P)_{1/3} = 2\eta L(9Q\sqrt{2})^{1/3}S^{-2} \tag{19}$$

We now consider the pressure drop in a tapering duct with entrance and exit sides and heights $S_1$,$H_1$ and $S_2$,$H_2$, respectively. Since in an equilateral triangle $S = 2H/\sqrt{3}$, eq. (17) becomes

$$\Delta P = 4\eta L\left(Q\cdot\frac{3n+1}{8n\sqrt{2}}\right)^n \sqrt{3}^{1-n)}\left(\frac{2H}{\sqrt{3}}\right)^{-(3n+1)} \tag{20}$$

The incremental pressure drop $dP$ is therefore given by

$$dP = 4\eta\, dl\left(Q\cdot\frac{3n+1}{8n\sqrt{2}}\right)^n \sqrt{3}^{(1-n)}\left(\frac{2h}{\sqrt{3}}\right)^{-(3n+1)} \tag{21}$$

where $h$ is the height of the channel at any point $l$ from the entrance. From trigonometry, $\tan\theta = dh/2\, dl$, and $dl$ is therefore

$$\mathrm{dl} = \frac{\mathrm{dh}}{2}\cot\theta \tag{22}$$

where $\theta$ is the taper angle. Substituting for $dl$ in eq. (21),

$$dP = 2\eta\, dh\cot\theta\left(Q\cdot\frac{3n+1}{8n\sqrt{2}}\right)^n(\sqrt{3})^{1-n}\left(\frac{2}{\sqrt{3}}\right)^{-(3n+1)}h^{-(3n+1)}$$

$$= 2^{-3n}\eta\cot\theta\left(Q\cdot\frac{3n+1}{8n\sqrt{2}}\right)^n 3^{n+1}h^{-(3n+1)}dh$$

Integrating between the limits of $H_1$ and $H_2$,

$$\Delta P = \eta\cot\theta\left(Q\cdot\frac{3n+1}{64n\sqrt{2}}\right)^n 3^{n+1}(H_2^{-3n} - H_1^{-3n})\left(-\frac{1}{3n}\right)$$

or

$$\Delta P = \frac{3^n n\cot\theta}{n}\left(Q\cdot\frac{3n+1}{64n\sqrt{2}}\right)^n(H_1^{-3n} - H_2^{-3n}) \tag{23}$$

In the Newtonian case ($n = 1$),

$$\Delta P = (\Delta P)_N = \frac{3\sqrt{2}}{32} Q\eta \cot\theta (H_1^{-3} - H_2^{-3}) \tag{24}$$

and for $n = 1/3$ (the common value for many polymer melts),

$$\Delta P = (\Delta P)_{1/3} = 3^{4/3}\eta \cot\theta \left(Q \cdot \frac{3\sqrt{2}}{32}\right)^{1/3} (H_1^{-1} - H_2^{-1})$$

or

$$(\Delta P)_{1/3} = \eta \cot\theta \left(Q \cdot \frac{243\sqrt{2}}{32}\right)^{1/3} (H_1^{-1} - H_2^{-1}) \tag{25}$$

Since $H = (S/2)\sqrt{3}$, eqs. (23), (24), and (25) may be rewritten in terms of $S_1$ and $S_2$, giving, respectively,

$$\Delta P = \frac{\eta \cot\theta}{n} \left(Q \cdot \frac{3n+1}{6n} \sqrt{6}\right)^n (S_1^{-3n} - S_2^{-3n}) \tag{26}$$

$$(\Delta P)_N = \frac{2}{3} \sqrt{3}\, \eta\, Q \cot\theta (S_1^{-3} - S_2^{-3}) \tag{27}$$

and

$$(\Delta P)_{1/3} = 3\eta \cot\theta (Q\sqrt{6})^{1/3} (S_1^{-n} - S_2^{-n}) \tag{28}$$

## DISCUSSION

This note continues a series of communications which consider viscous flow of power law liquids through parallel-sided and tapered channels with circular, wide-slit, arbitrarily (not necessarily *wide*-slit) rectangular, and regularly polygonal cross sections. A series of equations was derived which used the appropriate Rabinowitsch-corrected Hagen–Poiseuille equations. In the case of the polygonal channels a function $\lambda(N)$ was defined in conjunction with the assumption that the flow rate through a regular polygon is the same as that through a circle of equal cross-sectional area. This function compensates for the inaccuracies of the assumption which are seen to be severe in Newtonians in the case of the triangle, still substantial for the square, but quickly peter out thereafter. In typical pseudoplastics ($n \leqslant 0.4$) the error is negligible even when $N < 5$. The square geometry was analyzed in our preceding paper and the triangular geometry was dealt with in a semirigorous manner appropriate to it, so that we have two alternative equations each for the square and for the triangular channels. We are constructing suitable dies and will test the equations against the observed pressure drops. This will be reported in due course.

## SUMMARY

A general method for analyzing viscous flow through regular polygonal channels is presented, based upon the assumption that the flow rate through regular polygonal sections equals that through circular sections of the same area. In addition we offer alternatives which apply specifically to equilaterial triangles and to squares.

## References

1. R. S. Lenk and R. A. Frenkel, *J. Appl. Polym. Sci.*, (to appear).
2. F. N. Cogswell, private communication.

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